# Linear Response Theory and the Time-Dependent Ginzburg-Landau Equations*

A. Houghton† and K. Maki‡

*Service de Physique des Solides, Faculté des Sciences, 91 Orsay, France*

(Received 14 August 1970)

The transport properties associated with vortex motion in type-II superconductors are reformulated in terms of linear response theory (LRT). In particular, in the dirty limit we establish a simple relation between LRT and the time-dependent Ginzburg-Landau equation. In the more general case of a type-II superconductor with arbitrary electronic mean free path, we present the expressions for the transport coefficients in terms of retarded products. These expressions will be particularly useful in determining the transport properties of clean type-II superconductors.

## I. INTRODUCTION

Recently there has been a great deal of interest in transport phenomena associated with vortex motion in type-II superconductors in the mixed state, and type-I superconductors in the intermediate state.[1] In particular, these states exhibit a finite resistivity, and also a large transverse thermoelectric effect, if the electric field is applied perpendicular to the static magnetic field. This is in strong contrast to the behavior of superconductors in the Meissner state.

So far, two different theoretical approaches to this problem have been proposed: (i) a generalization of linear response theory to include the effects of dynamical fluctuations of the order parameter[2,3]; and (ii) methods which make use of the time-dependent Ginzburg-Landau (TDGL) equations.[4] However, the consistency of these two methods, as well as the region of validity of the simple TDGL, remains to be clarified. For example, in the case of the electromagnetic conductivity, the expressions obtained by the two methods did not agree in the dc limit. In particular, the expressions obtained by Caroli and Maki[2] (CM1) diverge in the dc limit, when the electric field is perpendicular to the static magnetic field. This is, of course, in flat contradiction of the experimental observation of resistive behavior in the vortex state.[5] This problem has recently been reexamined by Thompson,[3] who found that in this geometry certain terms had been omitted in CM1. When these terms were included, the correct dc limit was obtained – that is, the reactive part of the conductivity is exactly equal to zero. The dc limit of the absorptive part of the conductivity thus obtained still differed from the TDGL calculation by Caroli and Maki (CM2).[4] Thompson[3] was then able to show that if, in addition to the flux-flow conductivity of CM2, certain "anomalous" contributions to the intrinsic conductivity were taken into account, the two results agreed in the dc limit.

In this paper we present a general formulation of the transport properties of superconductors in the presence of fluctuations of the order parameter. In doing this we are able to clarify, in the dirty limit, the relation between linear response theory (LRT) and TDGL. In particular we give an explicit calculation of the Ettingshausen coefficient in LRT which is found to agree with the earlier TDGL calculation. This new formulation can be used in cases where simple TDGL is not applicable, for example pure type-II superconductors.

## II. GENERAL THEORY

We consider a situation in which the dc magnetic field $\vec{B}$ is in the $z$ direction and the dc electric field $\vec{E}$ in the $y$ direction. The field $\vec{E}$ can be described by either a vector potential $\delta\vec{A}$ or a scalar potential $\delta\phi$, at least in the resistive state. Then in LRT the change induced in a physical observable $W$, for example the electric current, by a heat current, in the presence of fluctuations of the order parameter, can be written

$$\begin{aligned}\delta_A\langle W\rangle_{\omega\vec{q}} &= \langle[W,j_\mu]\rangle_{\omega\vec{q}}\,\delta A_\mu(\omega,\vec{q}) \\ &\quad+\langle[W,\Psi^\dagger]\rangle_{\omega,\vec{q}}\,\delta\Delta(\omega,\vec{q})+\langle[W,\Psi]\rangle_{\omega,\vec{q}}\,\delta\Delta^\dagger(\omega,\vec{q}) \\ &=\{\langle[W,j_\mu]\rangle_{\omega,\vec{q}}+\langle[W,\Psi^\dagger]\rangle_{\omega,\vec{q}}\,L(\omega,\vec{q})\langle[\Psi,j_\mu]\rangle_{\omega,\vec{q}} \\ &\quad+\langle[W,\Psi]\rangle_{\omega,\vec{q}}\,L(\omega,\vec{q})\langle[\Psi^\dagger,j_\mu]\rangle_{\omega,\vec{q}}\}\,\delta A_\mu(\omega,\vec{q})\ . \end{aligned}\tag{1}$$

$L(\omega,\vec{q})$, the fluctuation propagator, is given by

$$L(\omega,\vec{q})=|g|/\{1-g\langle[\Psi,\Psi^\dagger]\rangle_{\omega,\vec{q}}\}\ ,\tag{2}$$

where $g$ is the BCS coupling constant and

$$\Psi^+(\vec{r},t)=\psi_\uparrow^\dagger(\vec{r},t)\psi_\downarrow^\dagger(\vec{r},t)\ .\tag{3}$$

In Eq. (1) the electric current operator $\vec{j}(\vec{r},t)$ is given by

$$\vec{j}(\vec{r},t)=e\,\frac{(\vec{\nabla}-\vec{\nabla}')}{2im}\sum_\alpha\psi_\alpha^\dagger(\vec{r}\,',t')\psi_\alpha(\vec{r},t)\bigg|_{\substack{\vec{r}\,'=\vec{r}\\ t'=t}}\ ,\tag{4}$$

and the thermal averages are taken in the absence of dynamical fluctuations.

In a similar way, if the electric field is expressed in terms of a scalar potential $\delta\phi$, then the change in $W$ is given by

$$\delta_\phi\langle W\rangle_{\omega,\vec q} = e\{\langle[W,n]\rangle_{\omega,\vec q} + \langle[W,\Psi^\dagger]\rangle_{\omega,\vec q}\, L(\omega,\vec q)\langle[\Psi,n]\rangle_{\omega,\vec q} + \langle[W,\Psi]\rangle_{\omega,\vec q}\, L(\omega,\vec q)\langle[\Psi^\dagger,n]\rangle_{\omega,\vec q}\}\,\delta\phi(\omega,\vec q)\ . \tag{5}$$

Here the density operator $n$ is given by

$$n(\vec r,t) = \sum_\alpha \psi^\dagger_\alpha(\vec r\,',t')\psi_\alpha(\vec r,t)\Big|_{\vec r\,'=\vec r,\ t'=t}\ . \tag{6}$$

Within the framework of LRT, these expressions are completely general. If we replace $W$ by the electrical current operator, the electrical conductivity is obtained; on the other hand, if $W$ is replaced by the heat current operator $\vec j^{\,h}_0$, where

$$\vec j^{\,h}_0 = -\frac{1}{2m}\sum_\alpha \vec\nabla\frac{\partial}{\partial t'} + \vec\nabla'\frac{\partial}{\partial t}\,\psi^\dagger_\alpha(\vec r\,',t')\psi_\alpha(\vec r,t)\Big|_{\vec r\,'=\vec r,\ t'=t}\ , \tag{7}$$

we obtain the heat flow.

In Sec. III we use these expressions in the dirty limit to establish the relationship between LRT and TDGL. In Sec. IV we give as a specific example the LRT calculation of the Ettingshausen coefficient.

### III. DIRTY LIMIT

We limit ourselves to the vicinity of $H_{c2}$ where the order parameter is small and given by the Abrikosov[6] solution

$$\Delta(\vec r) = \sum_n C_n e^{inky} e^{-eB(x-kn/2eB)^2} \tag{8}$$

and we can calculate it by expanding in powers of the order parameter. It is convenient to break up $\langle[W,j_\mu]\rangle_{\omega\vec q}$ into two parts $Q^{(1)}_{W,j_\mu}$ and $Q^{(2)}_{W,j_\mu}$ which we represent in Figs. 1 and 2, respectively. We write

$$\delta\langle W\rangle_A = (Q^{(1)}_{W,j_\mu} + Q^{(2)}_{W,j_\mu} + Q^{(3)}_{W,j_\mu})\,\delta A_\mu(\omega,\vec q)\ , \tag{9}$$

where $Q^{(3)}_{W,j_\mu}$, the contribution of the last two terms in Eq. (1) is represented diagrammatically in Fig.

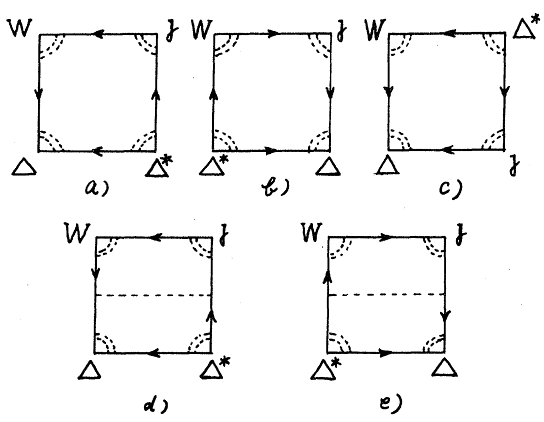


FIG. 1. Diagrams contributing to $Q^{(1)}$. These terms exist in the absence of fluctuations.

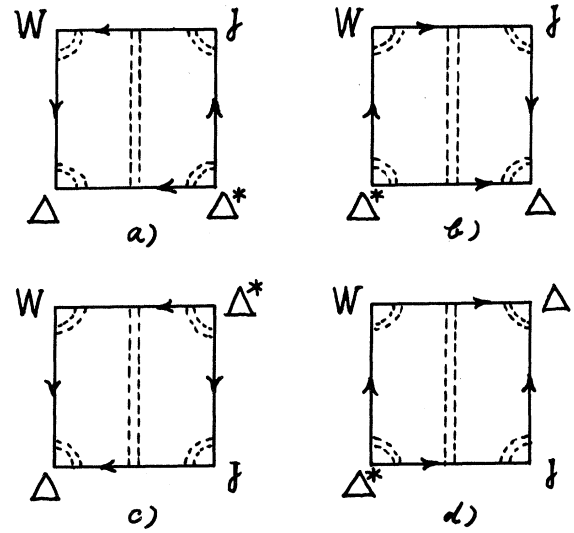


FIG. 2. Diagrams which contribute only in the geometry $\vec E \perp \vec B$. These terms contribute only when the external perturbation can couple the ground state of the order parameter to excited modes.

3. The contribution to $\delta\langle W\rangle_A$ arising from $Q^{(1)}_{W,j_\mu}$ is independent of the geometrical configuration of $\vec E$ and $\vec B$; on the other hand, the contribution from $Q^{(2)}_{W,j_\mu}$ only exists if $\vec E$ has a component perpendicular to $\vec B$. The physical origin of this term lies in the fact that in this geometry the external perturbation can couple the ground state of the order parameter to excited modes. Finally $Q^{(3)}_{W,j_\mu}$ involves propagation of the excited modes of the order parameter. As Thompson[3] has pointed out, only $Q^{(1)}_{W,j_\mu}$ and $Q^{(3)}_{W,j_\mu}$ were taken into account in the original CM 1 calculation.

In a similar way we may write

$$\delta\langle W\rangle_\phi = (Q^{(1)}_{W,\rho} + Q^{(2)}_{W,\rho} + Q^{(3)}_{W,\rho})\,\delta\phi(\omega,\vec q)\ . \tag{10}$$

In the dc limit the relation

$$\delta\langle W\rangle_A = \delta\langle W\rangle_\phi \tag{11}$$

follows from gauge invariance, provided we put

$$i\omega\delta\vec A = -\vec\nabla\delta\phi = \vec E \tag{12}$$

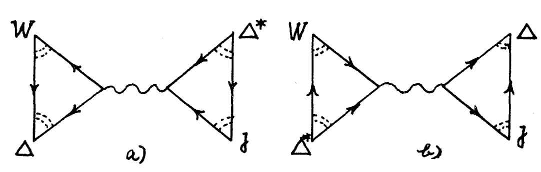


FIG. 3. Diagrams which involve propagation of fluctuations of the order parameter.

and $\vec{E}$ is in the plane perpendicular to $\vec{B}$. Therefore generally speaking we may evaluate the transport coefficients using either Eq. (9) or Eq. (10). This statement is of course applicable in the case of arbitrary mean free path. In the dirty limit Eq. (10) has a very simple relationship to TDGL; we will show that such a relation only exists if $Q_{W\rho}^{(2)}=0$.

In the Appendix we show explicitly that $Q_{W\rho}^{(2)}=0$ if $W$ is either $\vec{j}$ of $\vec{j}_0^h$. Therefore in these cases all the effects of fluctuations are contained in $Q_{W\rho}^{(3)}$. Further, it is easy to show that $Q_{j\rho}^{(3)}$ and $Q_{\vec{j}_0^h,\rho}^{(3)}$ reduce to the values calculated previously in CM 2.

In the specific case of the electric current we have

$$\delta\langle \vec{j}\rangle_A=(Q_{j\rho}^{(1)}+Q_{j\rho}^{(3)})\delta\phi \ , \tag{13}$$

where the term $Q_{j\rho}^{(1)}$, which arises from the intrinsic conductivity, can be shown to be given by

$$Q_{j\rho}^{(1)}=Q_{j\rho}^{(1)a}+Q(l/\xi_0) \ , \tag{14}$$

where $Q_{j\rho}^{(1)a}$ are the so-called anomalous terms discussed by Thompson.[3] In fact Thompson originally pointed out that if $Q_{j\rho}^{(1)}$ is added to the CM 2 result then the dc limit of the LRT calculation with a vector potential is obtained. This follows in the above formalism by comparing (13) and (14). We can therefore still describe the dc limit behavior of the resistive state in terms of TDGL if the original expression for the current[4] is slightly modified. The TDGL is now given by

$$\left(\frac{\partial}{\partial t}-2ie\phi\right)\Delta(\vec{r},t)=[D(\vec{\nabla}-2ie\vec{A})^2+\epsilon_i]\Delta(\vec{r},t) \ , \tag{15}$$

$$\vec{j}=\sigma_s\vec{E}+\frac{e\tau_{tr}N}{4\pi mT}\,i^{-1}(\vec{\nabla}_1-\vec{\nabla}_2-4ie\vec{A})\times\psi^{(1)}(\tfrac{1}{2}+\rho)\Delta(1)\Delta^\dagger_{(2)}\Big|_{1=2=(\vec{r},t)} \ , \tag{16}$$

where

$$\sigma_s=\lim\left(\frac{1}{i\omega}\,Q_{j,\rho}(i\omega)\right)\frac{\delta\phi}{E}=\sigma\left(1+\frac{\langle|\Delta|^2\rangle}{2\pi T}\left[\tfrac{1}{2}\rho^{-1}\psi^{(1)}(\tfrac{1}{2}+\rho)+\tfrac{1}{2}\psi^{(2)}(\tfrac{1}{2}+\rho)\right]\right) \ . \tag{17}$$

In Eqs. (15) and (16) all the notation is the same as CM 2; in particular $\psi^{(1)}$ and $\psi^{(2)}$ are the tri- and the tetragamma functions $\rho=\epsilon_0/4\pi T$ and $\epsilon_0=2eDH_{c2}(T)$.

It should be noted that a recent calculation by Ebisawa and Takayama[7] has shown that Eqs. (15) and (16) considered separately are only correct in the vicinity of the transition temperature ($T\cong T_{c0}$), and that, in general, both the order parameter and the current obey much more complicated equations containing higher-order differential operators. It is of importance to note, however, that this simple set of equations (15) and (16) (i.e., TDGL) results in the correct expression for the conductivity if both Eqs. (15) and (16) are used together. Further, by comparing the expression for the complex conductivity thus obtained with microscopic theory we can show that the set of equations (15) and (16) describe correctly the frequency-dependent phenomena if $\omega$ is sufficiently smaller than $\epsilon_0(0)(\cong T_{c0})$. Therefore, the TDGL determines completely the low-frequency behavior of dirty type-II superconductors. We also emphasize here that enormous conceptual economy is achieved by using TDGL, since it involves only the time-dependent order parameter $\Delta(\vec{r}\cdot t)$ as the dynamical entity, whereas in LRT we have to treat both electrons and order parameter on an equal footing.

Recent experiments in dirty superconductors[8–10] seem to be described quite well by only the TDGL terms (i.e., $Q_{j\rho}^{(3)}$). This is, we believe, because the experimental slope of the flux-flow resistivity as a function of the dc magnetic field $H$ is not taken in the immediate vicinity of $H_{c2}$ but in a region where the resistivity changes almost linearly with $H$. However, in this field range, the TDGL term completely dominates the flux-flow resistivity (at least in the dirty limit). $Q_{2\rho}^{(3)}$ is proportional to $|M|/B$ ($M$ is the magnetization and $B$ is the induction) and therefore increases very rapidly as $H$ decreases below $H_{c2}$, whereas the anomalous terms depend strongly on $H$ in the vicinity of $H_{c2}$ but always remain of the order of $\sigma$. Therefore in order to compare with the complete theory it is important to have a precise experimental determination of the slope near $H_{c2}$, where the contribution from the anomalous terms can become appreciable. However, experimentally the slope changes so rapidly in the immediate vicinity of $H_{c2}$ that an accurate comparison seems to be very difficult at present.

## IV. ETTINGSHAUSEN EFFECT

Making use of the results of Sec. III, we can also calculate the transverse entropy flow using LRT. We first note that it is easy to show that $Q_{\vec{j}_0^h}^{(1)}=0$; further, as we have shown in the Appendix, $Q_{\vec{j}_0^h}^{(2)}=0$. Therefore we anticipate that a LRT calculation – in a gauge where the electric field is introduced by a vector potential – should yield the result obtained previously by TDGL. This is verified below by a direct calculation.

If we consider again the geometry in which the dc magnetic field $\vec{B}$ is in the $z$ direction and the dc electric field in the $y$ direction, the transverse entropy flow is given by

$$\langle j_{0x}^h\rangle=\alpha_{xy}^{(1)}E_y \ . \tag{18}$$

The Ettingshausen coefficient $\alpha_{xy}^{(1)}$ can be simply expressed in terms of the retarded product of the heat-current operator $j_0^h$ and the electric current operator $j$ as

$$\alpha_{xy}^{(1)}=\lim_{\omega\to 0}\,(1/i\omega)\langle[j_{0x}^h\,,\,j_y]\rangle'_\omega \ , \tag{19}$$

where, making use of Eq. (1),

$$\langle[\vec{j}_0^h\vec{j}]\rangle'_\omega = \langle[\vec{j}_0^h, \vec{j}]\rangle + \langle[\vec{j}_0^h, \Psi^\dagger]\rangle_\omega L_\omega \langle[\Psi, \vec{j}]\rangle_\omega + \langle[\vec{j}_0^h, \Psi]\rangle_\omega L_\omega \langle[\Psi^\dagger, \vec{j}]\rangle_\omega . \qquad (20)$$

Using the prescription of Sec. III we break up $\langle[j_{0x}^h, j_y]\rangle_\omega$ into the two terms $Q^{(1)}_{j_{0x}^h, j_y}$ and $Q^{(2)}_{j_{0x}^h, j_y}$, corresponding to the diagrams in Figs. 1 and 2, respectively. It is easy, using symmetry, to see that the off-diagonal elements of $Q^{(1)}_{j_{0x}^h, j_y}$ are 0. It can also be shown that, if the energy dependence of the density of states is neglected, the diagonal components of $Q^{(1)}_{j_{0x}^h, j_x}$, which give rise to the thermoelectric power (or Peltier effect), vanish identically.

The term $Q^{(2)}_{j_{0x}^h, j_y}$ given in Fig. 2, as we pointed out in Sec. III, does not necessarily vanish when the electric current has a component perpendicular to $\vec{B}$. In fact direct calculation gives

$$Q^{(2)}_{j_{0x}^h, j_y} = \frac{\epsilon_0}{2\pi}\pi T\left(\sum_{n=0}^{\infty} + \sum_{n=-\infty}^{-\nu}\right)\left[(2\omega_n+\omega_\nu)\left(\frac{1}{(|\omega_n|+\frac{1}{2}\epsilon_0)^2} - \frac{1}{(|\omega_{n+\nu}|+\frac{1}{2}\epsilon_0)^2}\right)\frac{1}{|2\omega_n+\omega_\nu|+3\epsilon_0}\right], \qquad (21)$$

where $\omega_\nu$ and $\omega_n$ are the even and odd Matsubara frequencies, respectively. In deriving this result we have made use of the fact that

$$\vec{q}_x \Delta_0^*(\vec{q})\frac{1}{|2\omega_n+\omega_\nu| + Dq^2}\vec{q}_y\Delta_1(\vec{q}) = -\frac{\epsilon_0}{2Di}\frac{1}{|2\omega_n+\omega_\nu|+3\epsilon_0}|\Delta(\vec{q})|^2, \qquad (22)$$

where

$$\vec{q}_x = (\tfrac{1}{2})(\Pi^+ + \Pi^-), \quad \vec{q}_y = (1/2i)(\Pi^+ - \Pi^-), \qquad (23)$$

$$\Pi^\pm = \Pi_x \pm i\Pi_y, \quad \vec{\Pi} = (1/i)\vec{\nabla} - 2e\vec{A}_0, \qquad (24)$$

and $\vec{A}_0$ is the vector potential describing the static magnetic field. We remark that in the BCS state, for example, we would have $\vec{q}\Delta = 0$ and there would consequently be no contribution from the diagrams corresponding to $Q^{(2)}_{j_{0x}^h, j_y}$. It is of interest to note that the diagonal terms, for example $Q^{(2)}_{j_{0x}^h, j_x}$, coming from the same class of diagrams, vanish identically because of the exact cancellation of diagrams (a) and (b), and of diagrams (c) and (d).

If we now perform the summation over $n$ and make the analytical continuation $\omega_n \to -i\omega$, we obtain

$$Q^{(2)}_{j_{0x}^h, j_y}(-i\omega) = \frac{\sigma\epsilon_0}{2e}\left\{\frac{1}{2\pi T}\left[\frac{\epsilon_0 - i\omega}{2\epsilon_0+i\omega}\psi^{(1)}\left(\frac{1}{2} - \frac{i\omega}{2\pi T}+\rho\right) - \frac{\epsilon_0+i\omega}{2\epsilon_0 - i\omega}\psi^{(1)}(\tfrac{1}{2}+\rho)\right] + \frac{6\epsilon_0}{(2\epsilon_0 - i\omega)^2}\left[\psi\left(\frac{1}{2}-\frac{i\omega}{4\pi T}+3\rho\right) - \psi(\tfrac{1}{2}+\rho)\right] - \frac{6\epsilon_0}{(2\epsilon_0+i\omega)^2}\left[\psi\left(\frac{1}{2}-\frac{i\omega}{4\pi T}+3\rho\right) - \psi\left(\frac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)\right]\right\}. \qquad (25)$$

In the low-frequency limit, $Q^{(3)}_{j_{0x}^h, j_y}$ reduces to

$$Q^{(2)}_{j_{0x}^h, j_y} = \frac{\sigma}{e}i\omega\left\{\frac{3}{2\epsilon_0}[\psi(\tfrac{1}{2}+3\rho) - \psi(\tfrac{1}{2}+\rho)] - \frac{3}{4\pi T}\psi^{(1)}(\tfrac{1}{2}+\rho) - \frac{\epsilon_0}{4(2\pi T)^2}\psi^{(2)}(\tfrac{1}{2}+\rho)\right\}. \qquad (26)$$

Finally the contribution $Q^{(3)}_{j_{0x}^h, j_y}$ coming from the last two terms in Eq. (1) is represented in Fig. 3. The components of diagram (a) are (i) $\langle[j_{0x}^h, \Psi^\dagger]\rangle_\omega$, (ii) $L_\omega$, and (iii) $\langle[\Psi, j_y]\rangle_\omega$. We find

$$\langle[j_{0x}^h, \Psi^\dagger]\rangle_\omega = -N(0)D(\vec{q})\Delta(\vec{q})\left[\frac{\epsilon_0+i\omega}{2\epsilon_0 - i\omega}\psi(\tfrac{1}{2}+\rho) - \frac{\epsilon_0 - i\omega}{2\epsilon_0+i\omega}\psi\left(\frac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right) - \left(\frac{3\epsilon_0}{2\epsilon_0 - i\omega} - \frac{3\epsilon_0}{2\epsilon_0+i\omega}\right)\psi\left(\frac{1}{2}-\frac{i\omega}{4\pi T}+3\rho\right)\right], \qquad (27)$$

while $L(\omega)$ and $\langle[\Psi, j]\rangle$ have been calculated in CM 1 and are given by

$$L(\omega) = -N(0)^{-1}\left[\psi\left(\frac{1}{2}-\frac{i\omega}{4\pi T}+3\rho\right) - \psi(\tfrac{1}{2}+\rho)\right]^{-1}, \qquad (28a)$$

$$\langle[\Psi, j_x]\rangle = \frac{\sigma}{2e}\left\{(2\epsilon_0 - i\omega)^{-1}\left[\psi\left(\frac{1}{2}-\frac{i\omega}{4\pi T}+3\rho\right) - \psi(\tfrac{1}{2}+\rho)\right] + (2\epsilon_0+i\omega)^{-1}\left[\psi\left(\frac{1}{2}-\frac{i\omega}{4\pi T}+3\rho\right) - \psi\left(\frac{1}{2}-\frac{i\omega}{2\pi T}+\rho\right)\right]\right\}2\vec{q}_x\Delta_0\vec{q}. \qquad (28b)$$

Combining these factors and adding diagram 3(b) we obtain in the low-frequency limit

$$Q^{(3)}_{j^h_{0x},j_y}=\frac{\sigma}{e}\frac{i\omega}{\epsilon_0}\left\{\tfrac{3}{2}[\psi(\tfrac{1}{2}+3\rho)-\psi(\tfrac{1}{2}+\rho)]-\rho\psi^{(1)}(\tfrac{1}{2}+\rho)\right\}. \tag{29}$$

In deriving this equation we have made use of the relation

$$(\vec{q}_x\Delta^*)(q_y\Delta)=(\epsilon_0/2iD)|\Delta|^2\,. \tag{30}$$

It should be noted that the diagonal part of the fluctuation contribution $Q^{(3)}_{j^h_{0x},j_x}$ (or $Q^{(3)}_{j^h_{0y},j_y}$) vanishes owing to the exact cancellation between diagrams 3(a) and 3(b), as in the case of $Q^{(2)}_{j^h_{0x},j_x}$ (or $Q^{(2)}_{j^h_{0y},j_y}$).

The Ettingshausen coefficient is now obtained from the sum of Eqs. (26) and (29) as

$$\alpha^{(1)}_{xy}=-\frac{\sigma}{e}\frac{|\Delta|^2}{2\pi T}\psi^{(1)}(\tfrac{1}{2}+\rho)\left[1+\frac{\rho}{2}\frac{\psi^{(2)}(\frac{1}{2}+\rho)}{\psi^{(1)}(\frac{1}{2}+\rho)}\right], \tag{31}$$

which agrees with the previous calculation of the constant by TDGL.[4] The same expression has very recently been derived by Ebisawa and Takayama[7] who have chosen the gauge $\vec{\nabla}\delta\phi=-\vec{E}$.

We should point out that Eq. (31) only contains the electronic contribution to the Ettingshausen coefficient. However, as pointed by one of us (KM[11]), when the magnitization is time dependent there is a further contribution to the Ettingshausen effect associated with the magnitization current. If this term is added, we obtain the complete Ettingshausen effect

$$\alpha_{xy}=-\frac{\sigma}{e}\frac{|\Delta|^2}{4\pi T}\psi^{(1)}(\tfrac{1}{2}+\rho)\left[1+\rho\frac{\psi^{(2)}(\frac{1}{2}+\rho)}{\psi^{(1)}(\frac{1}{2}+\rho)}\right]. \tag{32}$$

## V. CONCLUSION

In this paper we have used LRT to obtain general expressions for the transport coefficients of the resistive state. In dirty superconductors, the dc limit of these results was shown to be equivalent to the predictions of simple TDGL. Therefore in the dirty limit, where TDGL is given by a set of differential equations, TDGL is a simple flexible scheme which can be used to determine those various physical phenomena in which motion of the order parameter is important. On the other hand, in situations where expansion in powers of the order parameter is not valid, TDGL itself is given by a set of very complicated integral equations. In this case the most straightforward approach is to use the exact expressions of LRT to determine the transport coefficients. We propose in a future publication to use this approach to determine the transport properties of clean type-II superconductors.

## ACKNOWLEDGMENTS

It is a great pleasure for the authors to thank the members of Service de Physique des Solides, Orsay, for their hospitality during the period of time in which this work was done.

## APPENDIX: DEMONSTRATION THAT $Q^{(2)}_{W\rho}=0$ FOR $W=j$ AND $W=j^h_0$

We assume, as in the text, that the dc magnetic field and the dc electric field are applied in the $z$ and $x$ directions, respectively.

Let us first consider the case where $W=\vec{j}$, the electric current. The diagrams contained in $Q^{(2)}_{jy}$, which are of the type shown in Fig. 2, are easily calculated if we note that they are equivalent to $Q^{(2)}_{jj}$ when the loop containing the vertex $j_\mu\delta A_\mu$ is replaced by $\rho\delta\phi$. More precisely, $Q^{(2)}_{j\rho}$ is simply obtained from $Q^{(2)}_{jj}$ by replacing $2iDq_\mu\delta A_\mu$ by $\omega_n/|\omega_n|\delta\phi$ before evaluating the frequency sum ($\omega_n$ is the internal frequency of the electron loop). If we compare with Thompson's calculation of $Q^{(2)}_{jj}$ [we note that there is a factor-of-2 error in his Eq. (9)], we find that

$$Q^{(2)}_{j_\mu\rho}=\sigma(q_\mu\Delta^\dagger)(\delta\phi\Delta)\,4i\pi T\left(\sum_{n=0}^{\infty}-\sum_{-\infty}^{n=-\nu}\right)\left(\frac{1}{2|\omega_{n+\nu}|+\epsilon_0}+\frac{1}{2|\omega_n|+\epsilon_0}\right)^2\frac{1}{|\omega_n+\omega_{n+\nu}|+3\epsilon_0}\,. \tag{A1}$$

We have assumed that $j$ is perpendicular to the dc magnetic field. From the form of Eq. (A1), it is easy to see that $Q^{(2)}_{j_\mu\rho}$ vanishes identically. The cancellation between the two sums in (A1) has its origin in the additional frequence $\omega_n/|\omega_n|$ mentioned above.

When $W=j^h_0$, it is easy to show that $Q^{(2)}_{j^h_\mu\rho}$ vanishes by making use of the same replacement in $Q^{(2)}_{j^h_\mu,j_\nu}$ given in the text.

The cancellation discussed above occurs because the quantity in question takes a different sign in the frequency ranges $\omega_n>0$ and $\omega_n+\nu<0$. Rearranging the function under the summation signs, it is easy to show that these two terms exactly cancel. It is important to note that, although the present proof was only carried out in the dirty limit, the conclusion that $Q^{(2)}_{W\rho}\equiv 0$ (for $W=j$ and $W=j^h_0$) is independent of the electronic mean free path of the system under consideration.

*Work supported in part by the Advanced Research Projects Agency and the National Science Foundation.

†Permanent address: Physics Department, Brown University, Providence, R. I.

‡Permanent address: Department of Physics, Tohoku University, Sendai, Japan.

[1]See for a comprehensive review Y. B. Kim and M. J. Stephen, in *Treatise on Superconductivity*, edited by R. D. Parks (Marcel Dekker, New York, 1969).

[2]C. Caroli and K. Maki, Phys. Rev. 159, 306 (1967).

[3]R. S. Thompson, Phys. Rev. B 1, 327 (1970).

[4]C. Caroli and K. Maki, Phys. Rev. 164, 591 (1967).

[5]See, for example, Y. B. Kim, C. F. Hampstead, and A. R. Strnad, Phys. Rev. 129, 523 (1963).

[6]A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. 32, 1442 (1957) [Soviet Phys. JETP 5, 1174 (1957)].

[7]H. Ebisawa and H. Takayama, Progr. Theoret. Phys. (Kyoto) 42, 1481 (1969); and to be published.

[8]J. A. Cape and I. F. Silvera, Phys. Rev. Letters 20, 326 (1968).

[9]Y. Muto, K. Mori, and K. Noto, Proceedings on the International Conference on the Science of Superconductivity, Stanford, 1969 (to be published in Physica).

[10]G. Fischer, R. D. McConnell, P. Monceau, and K. Maki, Phys. Rev. B 1, 2134 (1970).

[11]K. Maki, Phys. Rev. Letters 21, 1755 (1968).